Fundamental functions:

1. **Linear function: y=ax+b**; $a,b \in R$

Such a function can be written as

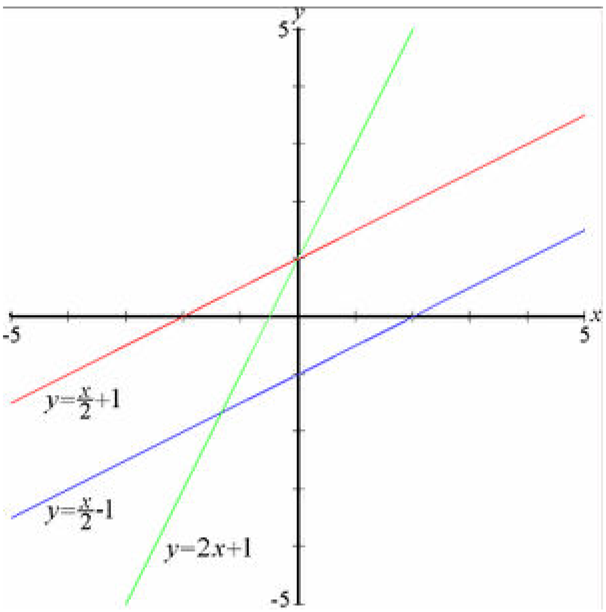
$$f(x) = mx + b$$

(called slope-intercept form), where m and b are real constants and x is a real variable. The constant m is often called the slope, while b is the y-intercept, which gives the point of intersection between the graph of the function and the y-axis. Changing m makes the line steeper or shallower, while changing b moves the line up or down.

Examples of functions whose graph is a line include the following:

- $f_1(x) = 2x + 1$
- $f_2(x) = x / 2 + 1$
- $f_3(x) = x / 2 - 1$.

The graphs of these are shown in the image:



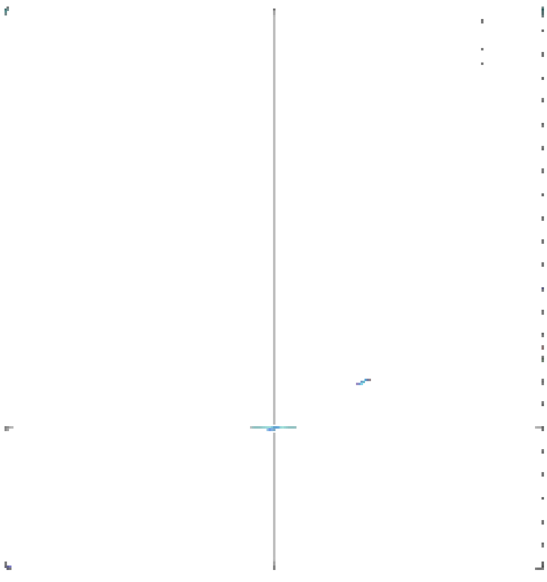
2. **Power function: y=x^a**; $a \in R, a \neq 0, a \neq 1$

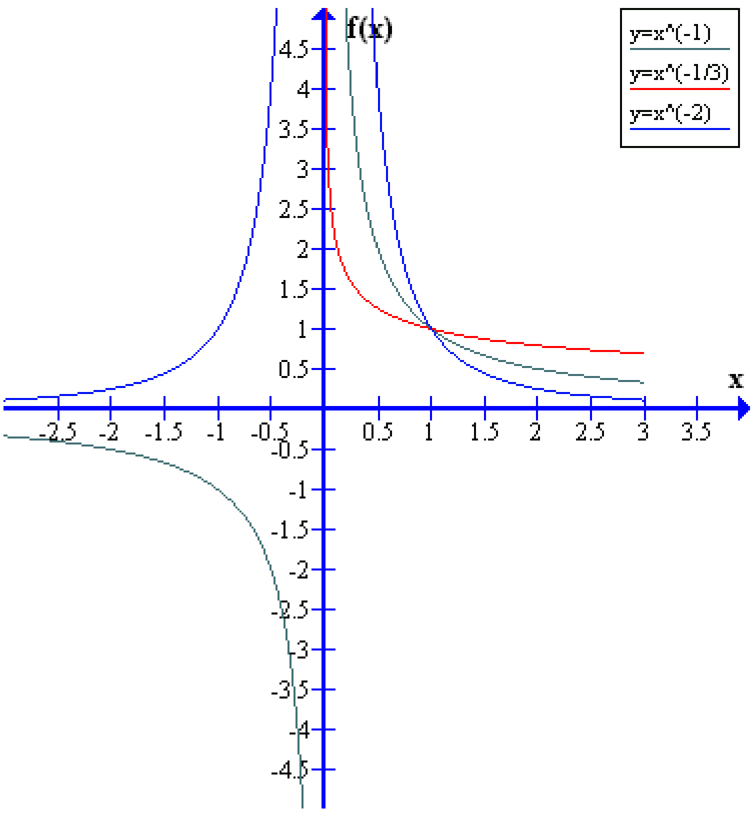
Power function has different properties for different exponents a:

- a is a natural number – the domain of such functions is set of real numbers;
- a is a integer negative – the domain of such functions is set of real numbers without 0;

- *a* is a fraction *1/m* where *m* is a natural odd number – the domain of such functions is set of real numbers;
- *a* is a fraction *1/m* where *m* is a natural even number – the domain of such functions is set of real positive numbers;

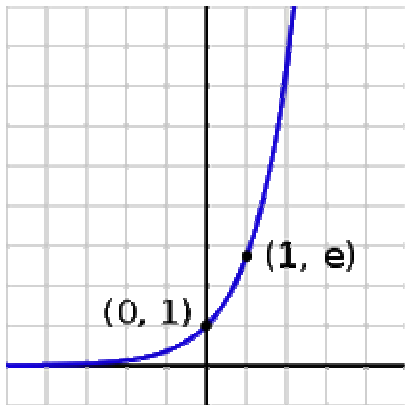
Examples of plots power functions with different *a*.



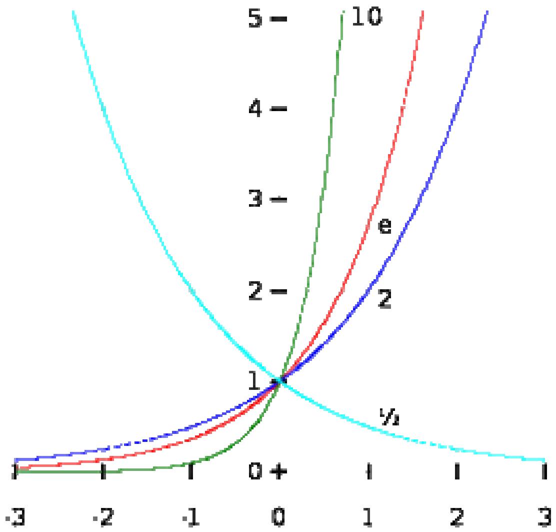


3. **Exponential function: $y=a^x$;** $a\in R$, $a>0$

As a function of the *real* variable x, the graph of $y = a^x$ is always positive (above the x axis) and increasing. It never touches the x axis, although it gets arbitrarily close to it (thus, the x axis is a horizontal asymptote to the graph).
Special example of exponential function is $y = e^x$ with base e≈2,7182818285… (e is irrational number and $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$)



$y=e^x$



Exponentiation with various bases; from top to bottom, base 10 ($y=10^x$), base e ($y=e^x$), base 2 ($y=2^x$), base ½ ($y=(1/2)^x$). Note how all of the curves pass through the point (0, 1). This is because, in accordance with the properties of exponentiation, any nonzero number raised to the power 0 is 1. Also note that at x=1, the y value equals the base. This is because any number raised to the power 1 is that same number.

4. **Logarithmic function:** $y=\log_a x$; $a\in R$, $a>0$

Logarithmic function is the inverse function to exponential function $y=b^x$ and is defined for all positive x.

The **logarithm** of a number to a given base is the power to which the base must be raised in order to produce the number. So, for a number x, a base a and an exponent y:

$$y = \log_a x \Leftrightarrow x = a^y$$

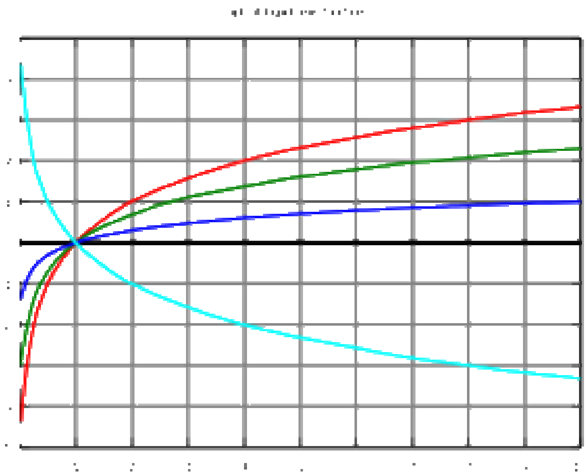
The **natural logarithm**, ln(x), is the logarithm of a number x to the base e. So,

$$\ln x = \log_e x$$

The **common logarithm**, log(x), is the logarithm of a number x to the base *10*. So,

$$\log x = \log_{10} x$$

Plots of logarithm to different bases.



Logarithm functions, graphed for various bases: red is to base *2*, green is to base e, blue is to base 10 and cyan is to base 1/2. Each tick on the axes is one unit. Logarithms of all bases pass through the point (1, 0), because any number raised to the power 0 is 1, and through the points (a, 1) for base a, because a number raised to the power 1 is itself. The curves approach the y-axis but do not reach it because of the singularity at $x = 0$.

Properties of the logarithm

a. The major property of logarithms is that they map multiplication to addition. This ability stems from the following identity:

$$b^x \times b^y = b^{x+y} \ ,$$

which by taking logarithms becomes

$$\log_b\left(b^x \times b^y\right) = \log_b\left(b^{x+y}\right) = x + y = \log_b\left(b^x\right) + \log_b\left(b^y\right) .$$

b. A related property is reduction of exponentiation to multiplication. Using the identity:

$$c = b^{\log_b(c)} \quad ,$$

it follows that c to the power p (exponentiation) is:

$$c^p = \left(b^{\log_b(c)}\right)^p = b^{p\log_b(c)} \quad ,$$

or, taking logarithms:

$$\log_b\left(c^p\right) = p\log_b(c).$$

So, we can also write:

$$a^{\log_a b} = b,$$
$$\log_a 1 = 0,$$
$$\log_a a = 1.$$
$$\log_a(b\cdot c) = \log_a b + \log_a c,$$
$$\log_a \tfrac{b}{c} = \log_a b - \log_a c,$$
$$\log_a b^c = c\cdot\log_a b,$$
$$\log_a \sqrt[n]{b^c} = \tfrac{c}{n}\log_a b,$$
$$\log_{a^n} b = \tfrac{1}{n}\log_a b,$$
$$\log_a b = \frac{1}{\log_b a},$$
$$\log_a b\cdot\log_b a = 1,$$
$$\ln 10\cdot\log e = 1.$$
$$\frac{\log_b x}{\log_b a} = \log_a x$$
$$\log_b x = \log_b a \log_a x$$
$$a^{log_c b} = b^{log_c a}$$

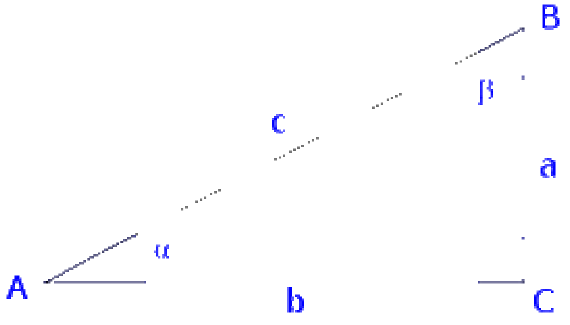
If the base is $a > 1$ then:

$$\lim_{x\to 0}\log_a x = -\infty$$
$$\lim_{x\to+\infty}\log_a x = +\infty$$

If the base is $0 < a < 1$ then:

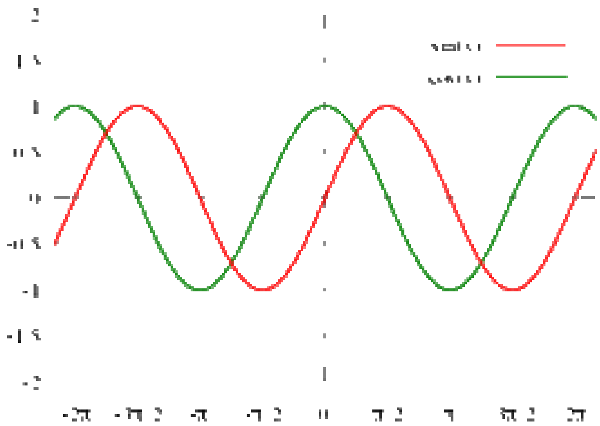
$$\lim_{x\to 0}\log_a x = +\infty$$
$$\lim_{x\to+\infty}\log_a x = -\infty$$

5. **Trigonometric functions**: y=sinx, y=cosx, y=tgx, y=ctgx;

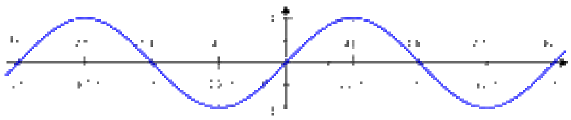


	$\frac{a}{\cdot}$	$\frac{b}{\cdot}$	$\frac{c}{\cdot}$
$\frac{\cdot}{a}$	1	$\operatorname{ctg}\ \alpha$	$\csc\ \alpha$
$\frac{\cdot}{b}$	$\operatorname{tg}\ \alpha$	1	$\sec\ \alpha$
$\frac{\cdot}{c}$	$\sin\ \alpha$	$\cos\ \alpha$	1

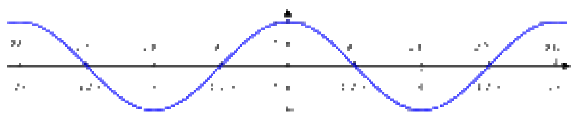
Function	Abbreviation	Identities (using radians)
Sine	sin	$\sin\theta \equiv \cos\left(\frac{\pi}{2}-\theta\right) \equiv \frac{1}{\csc\theta}$
Cosine	cos	$\cos\theta \equiv \sin\left(\frac{\pi}{2}-\theta\right) \equiv \frac{1}{\sec\theta}$
Tangent	tan (or tg)	$\tan\theta \equiv \frac{\sin\theta}{\cos\theta} \equiv \cot\left(\frac{\pi}{2}-\theta\right) \equiv \frac{1}{\cot\theta}$
Cotangent	cot (or ctg or ctn)	$\cot\theta \equiv \frac{\cos\theta}{\sin\theta} \equiv \tan\left(\frac{\pi}{2}-\theta\right) \equiv \frac{1}{\tan\theta}$



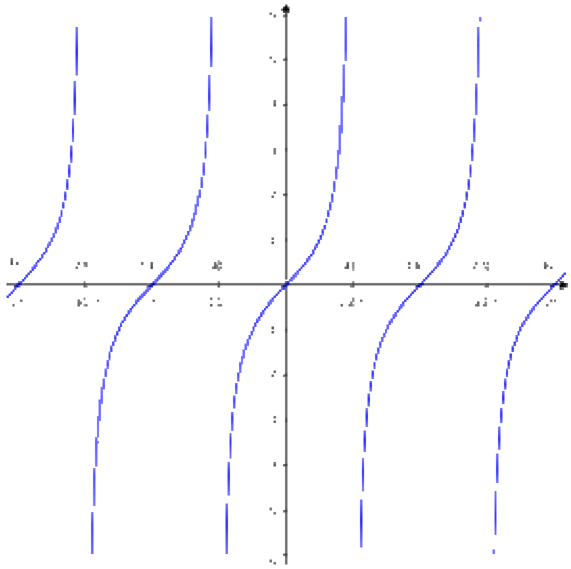
The sine and cosine functions graphed on the Cartesian plane.



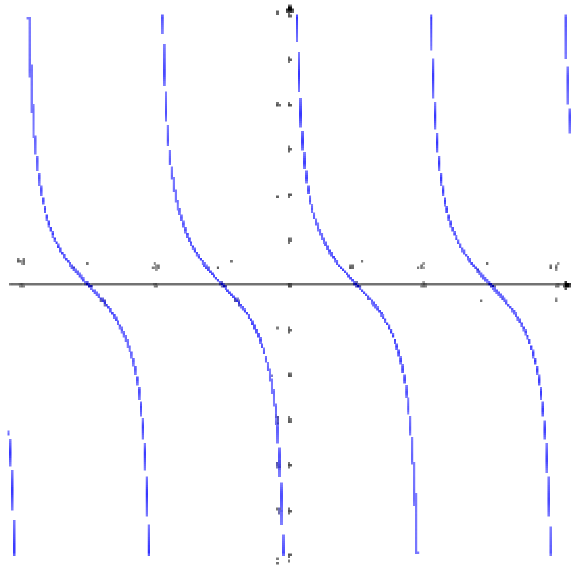
Sinusoid: $y = \sin x$



Cosinusoid: $y = \cos x$



Tangensoid: $y = \operatorname{tg}\ x$



Cotangensoid: $y = \operatorname{ctg}\ x$

radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
degrees	0°	30°	45°	60°	90°

sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tg	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	indeterminate
ctg	indeterminate	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

The trigonometric functions satisfy identities:

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

Angle sum identities

Sine	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
Cosine	$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
Tangent	$\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Double-angle formulas			
$\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$	$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$	$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$	$\cot 2\theta = \dfrac{\cot^2\theta - 1}{2\cot\theta}$

Power-reduction formulas

Sine	Cosine
$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$	$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

Product-to-sum identities

$$\cos\theta\cos\varphi = \frac{\cos(\theta-\varphi)+\cos(\theta+\varphi)}{2}$$
$$\sin\theta\sin\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{2}$$
$$\sin\theta\cos\varphi = \frac{\sin(\theta+\varphi)+\sin(\theta-\varphi)}{2}$$
$$\cos\theta\sin\varphi = \frac{\sin(\theta+\varphi)-\sin(\theta-\varphi)}{2}$$

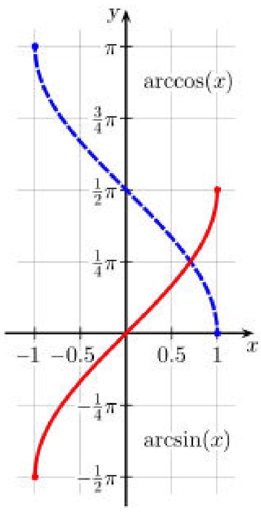
Sum-to-product identities

$\sin\theta+\sin\varphi = 2\sin\left(\dfrac{\theta+\varphi}{2}\right)\cos\left(\dfrac{\theta-\varphi}{2}\right)$
$\cos\theta+\cos\varphi = 2\cos\left(\dfrac{\theta+\varphi}{2}\right)\cos\left(\dfrac{\theta-\varphi}{2}\right)$
$\cos\theta-\cos\varphi = -2\sin\left(\dfrac{\theta+\varphi}{2}\right)\sin\left(\dfrac{\theta-\varphi}{2}\right)$
$\sin\theta-\sin\varphi = 2\cos\left(\dfrac{\theta+\varphi}{2}\right)\sin\left(\dfrac{\theta-\varphi}{2}\right)$

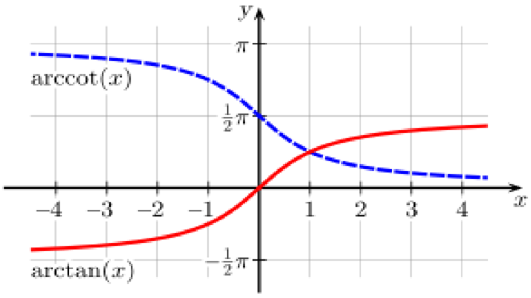
6. Cyclometric functions (inverse trigonometric functions): y=arcsinx, y=arccosx, y=arctgx, y=arcctgx

The **inverse trigonometric functions** or **cyclometric functions** are the so-called inverse functions of the trigonometric functions, though they do not meet the official definition for inverse functions as their domains are subsets of the images of the original functions. The principal inverses are listed in the following table.

Name	Usual notation	Definition	Domain of x for real result	Range of usual principal value
arcsine	$y = \arcsin(x)$	$x=\sin(y)$	−1 to +1	$-\pi/2 \le y \le \pi/2$
arccosine	$y = \arccos(x)$	$x=\cos(y)$	−1 to +1	$0 \le y \le \pi$
arctangent	$y = \mathrm{arctg}(x)$	$x = \mathrm{tg}(y)$	all	$-\pi/2 < y < \pi/2$
arccotangent	$y = \mathrm{arcctg}(x)$	$x =\mathrm{ctg}(y)$	all	$0 < y < \pi$



The usual principal values of the $f(x) = \arcsin(x)$ and $f(x) = \arccos(x)$ functions graphed on the cartesian plane.



The usual principal values of the $f(x)$ = arctg(x) and $f(x)$ = arcctg(x) functions graphed on the cartesian plane.

7. Hyperbolic functions: y=sinhx, y=coshx, y=tghx, y=ctghx.

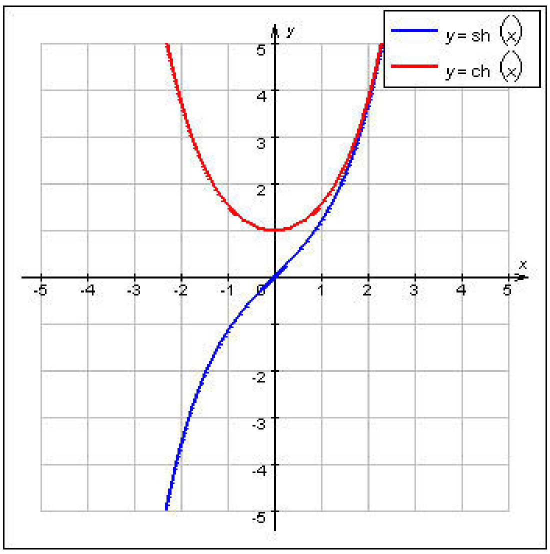
The hyperbolic functions are:

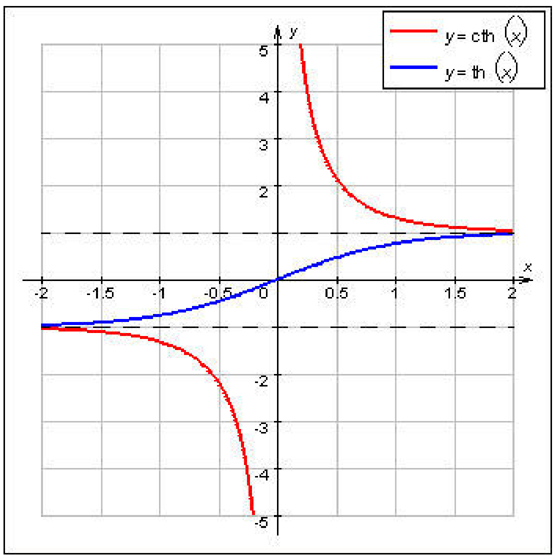
- Hyperbolic sine (sinhx or shx): $$\sinh x = \frac{e^x - e^{-x}}{2} = -i \sin ix$$
- Hyperbolic cosine (coshx or chx): $$\cosh x = \frac{e^x + e^{-x}}{2} = \cos ix$$
- Hyperbolic tangent (tghx or thx or tanhx):

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = -i \tan ix$$

- Hyperbolic cotangent (ctghx or cthx or cothx):

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{\frac{e^x + e^{-x}}{2}}{\frac{e^x - e^{-x}}{2}} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1} = i \cot ix$$





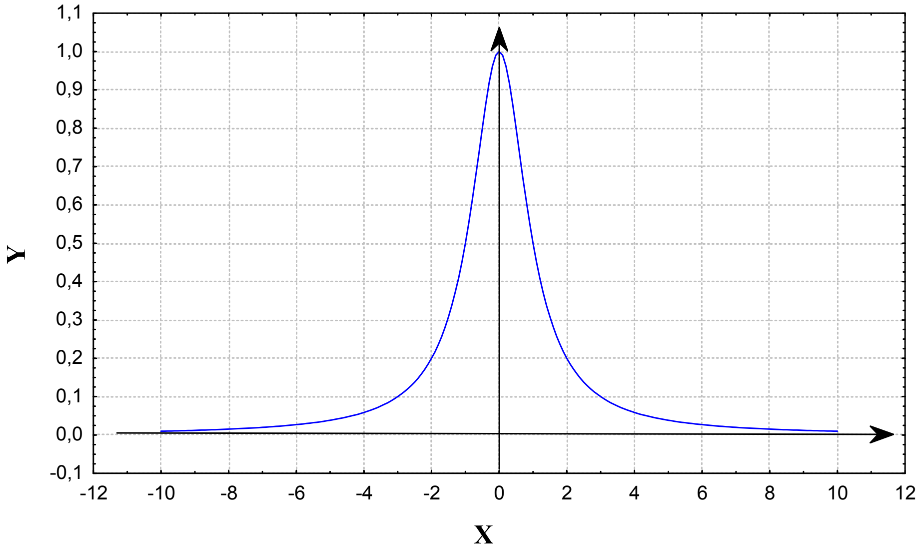
The hyperbolic functions satisfy identities (similar in form to the trigonometric identities):

$$\cosh^2 t - \sinh^2 t = 1$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$
$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$$
$$\cosh^2 \frac{x}{2} = \frac{\cosh x + 1}{2}$$
$$\sinh^2 \frac{x}{2} = \frac{\cosh x - 1}{2}$$
$$e^x = \cosh x + \sinh x$$
$$e^{-x} = \cosh x - \sinh x.$$

Function $y = \dfrac{a^2}{b^2 + x^2}$

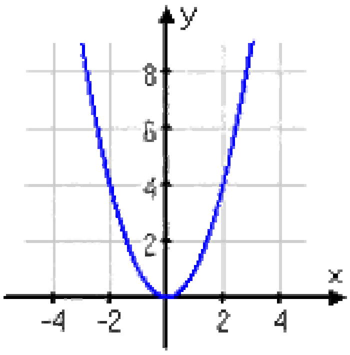
$D_f = \boldsymbol{R}; \quad f(D_f) = (0; \frac{a^2}{b^2}]$

Graph of the function y=1/(1+x^2)



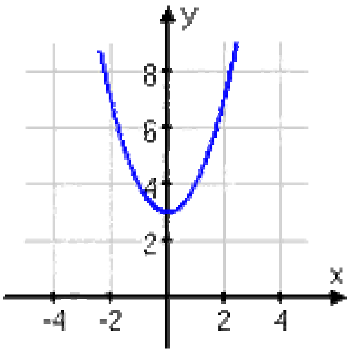
Transformations of Graphs of Functions

As an example we start with the basic quadratic function (a>0)



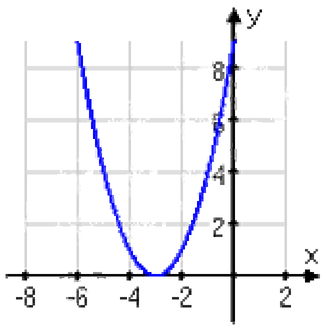
$y=x^2$

1. $f(x) + a$ – the graph of $f(x)$ is shifted upward a units; $f(x) - a$ – the graph of $f(x)$ is shifted downward a units



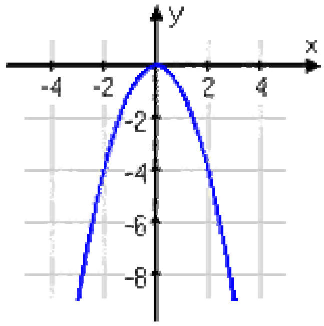
$y=x^2+3$

2. $f(x + a)$ is $f(x)$ shifted left a units; $f(x - a)$ – the graph of $f(x)$ is shifted right a units



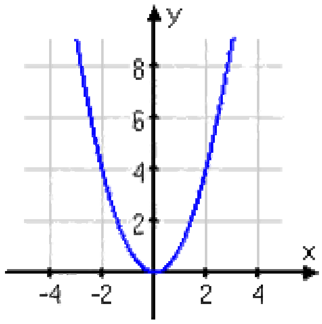
$y=(x+3)^2$

3. $-f(x)$ – the graph of $f(x)$ is flipped upside down ("reflected about the x-axis")



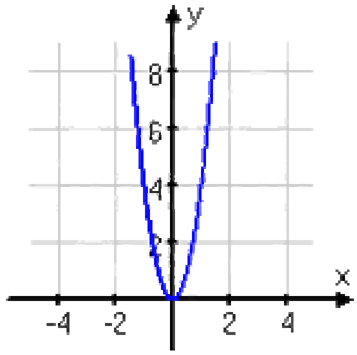
$y = -x^2$

4. $f(-x)$ – the mirror of the graph of $f(x)$ ("reflected about the y-axis")



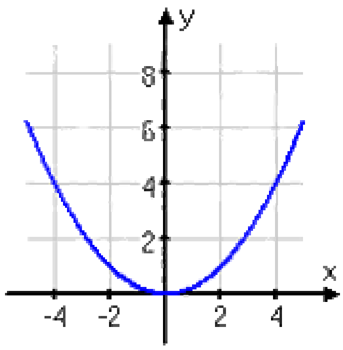
$y = (-x)^2$

5. $f(ax)$ $a>1$ scale x-axis is a times smaller



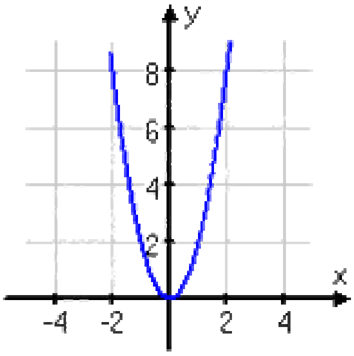
$y = (2x)^2$

6. $f(ax)$ $0<a<1$ scale x-axis is a times greater



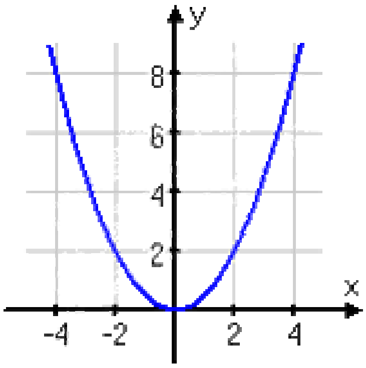
$y = ((1/2)x)^2$

7. $af(x)$ $a>1$ scale y-axis is a times greater



$y=2x^2$

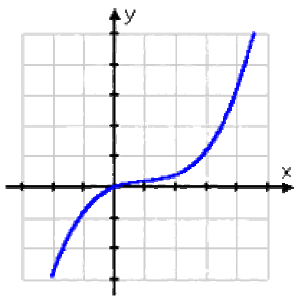
8. $af(x)$ $0<a<1$ scale y-axis is a times smaller



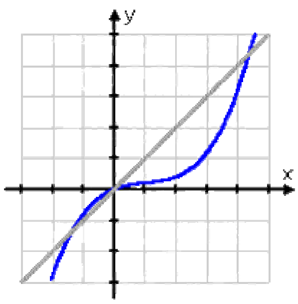
$y=(1/2)x^2$

The graph of a inverse function (only for bijective map)

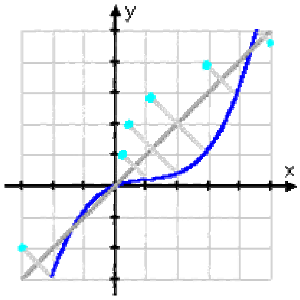
1. Suppose we have graph of function:



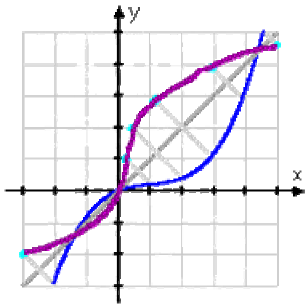
2. Now we draw the reflection line:



3. Now we make the reflection of each point about the bisector line:



4. Now we connect the dots. This new line is the graph of an inverse function.



Note that the points actually ON the line y = x (the bisector line) don't move; that is, where the function crosses the diagonal, the inverse will cross it, too.