LIST 9. Vector Spaces. Eigenvalues and Eigenvectors of Matrix.

1. Check if sets V, with addition two elements $a \in V$ i $b \in V$ and multiplication any element $a \in V$ by $\alpha \in K$ (K – algebraic field) satisfy conditions of vector space:
   a) V – set of all two dimensional vectors with integer coefficients with addition a+b and multiplication $\alpha \cdot a$?
   b) V – set of all vectors lying on the same straight line with addition a+b and multiplication $\alpha \cdot a$?
   c) V – set of all two dimensional vectors lying on the X-axis or Y-axis with addition a+b and multiplication $\alpha \cdot a$?
   d) V – set of all continuous functions a=f(x), b=g(x), $f: [0;1] \to R$, $g: [0;1] \to R$ with addition f(x)+g(x) and multiplication $\alpha \cdot f(x)$?
   e) V – set of all functions a=f(x), b=g(x), $f: R \to R^+$, $g: R \to R^+$ with addition $f(x) \cdot g(x)$ and multiplication $f^{\alpha}(x)$?
   f) V – set of all even functions a=f(x), b=g(x), with addition $f(x) \cdot g(x)$ and multiplication $\alpha \cdot f(x)$?
   g) V – set of all square matrices $a=[a_{ij}]$, $b=[b_{ij}]$ dimension $n \times n$, with addition $a+b=[a_{ij}+b_{ij}]$ and multiplication $\alpha a=[\alpha \cdot a_{ij}]$?
   h) V – set of all square non-singular matrices $a=[a_{ij}]$, $b=[b_{ij}]$ dimension $n \times n$, with addition $a+b=[a_{ij}] \cdot [b_{ij}]$ and multiplication $\alpha a=[\alpha \cdot a_{ij}]$?

2. Which of systems of vectors below are linearly independent?
   a) $w_1=(1,0)$, $w_2=(0,1)$ in R^2; b) $w_1=(1,2)$, $w_2=(2,4)$ in R^2;
   c) $w_1=(1,2)$, $w_2=(2,2)$ in R^2; d) $w_1=(i,0)$, $w_2=(1+i,2)$ in C^2.

3. Prove that in R^3 vectors (1,2,3), (3,2,1), (4,4,5) are linearly independent.

4. Find such a number *a* that in R^4 vectors (1,2,3,1), (0,3,-1,2), (1,0,3,-4), (2,5,a,-1) are linearly independent.

5. Which systems of vectors v_1, v_2, v_3 can be a basis in R^3?
   a) $v_1=(1,0,0)$, $v_2=(1,1,0)$, $v_3=(3,-1,1)$;
   b) $v_1=(3,1,2)$, $v_2=(2,1,2)$, $v_3=(-1,2,5)$;
   c) $v_1=(1,1,0)$, $v_2=(0,1,1)$, $v_3=(1,2,1)$.

6. Check if set V_3 is linear subspace of R^3 space:
$$V_3=\{(x_1,x_2,x_3) \in R^3 : x_1+x_2+x_3 \geq 0\}.$$

7. Find the dimension and bases of subspaces:
   a) $V_1=\{(x_1,x_2,x_3)\in R^3: x_1+x_2+x_3=0\}$;
   b) $V_2=\{(x_1,x_2,x_3)\in R^3: x_1=-2x_2,\ x_3=x_2\}$;
   c) $V_3=\{(x_1,x_2,x_3,x_4)\in R^4: x_3+x_4=0\}$;
   d) $V_4=\{(x_1,x_2,x_3,x_4)\in R^4: x_1+2x_2-x_4=0,\ 2x_3-x_4=0\}$.

8. Check if $w=(2,0,-1)$ belongs to $\mathrm{span}(v_1,v_2)$. If yes find its coordinates with respect to this basis.
   a) $v_1=(1,0,0)$, $v_2=(1,0,1)$; b) $v_1=(0,0,1)$, $v_2=(1,1,0)$.

9. Calculate eigenvalues and eigenvectors of the matrices:

$$a)\ \begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix};\quad b)\ \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix};\quad c)\ \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix};\quad d)\ \begin{bmatrix} 3 & -2 & 2 \\ 0 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix}.$$